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ON SIMPLICIAL AND CUBICAL COMPLEXES WITH SHORT LINKS

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ABSTRACT

We consider closed simplicial and cubical n-complexes in terms of the links of their $(n-2)$-faces. Especially, we consider the case when this

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link has size 3 or 4, i.e., every $(n-2)$-face is contained in 3 or 4 n-faces. Such simplicial complexes with *short* (i.e., of length 3 or 4) links are completely classified by their *characteristic partition*. We consider also embedding into (the skeletons of) hypercubes of the skeletons of simplicial and cubical complexes.

1. Introduction

An **n-dimensional simplicial complex** (or **simplicial n-complex**) is a collection $\mathcal{S}$ of finite nonempty sets, such that:

(i) if S is an element of $\mathcal{S}$, then so is every nonempty subset of S;

(ii) If $S, S' \in \mathcal{S}$, then $S \cap S' \in \mathcal{S}$;

(iii) all maximal (for inclusion) elements of $\mathcal{S}$ have cardinality $n+1$.

Given a simplicial complex $\mathcal{K}$ of dimension n, every $(k+1)$-subset of it defines a **face of dimension k**. In the sequel we identify faces with their set of vertices. A simplicial complex $\mathcal{K}$ is called a **pseudomanifold** if every $(n-1)$-face belongs to one or two n-faces. The **boundary** is the set of $(n-1)$-faces contained in exactly one n-face. If the boundary of $\mathcal{K}$ is empty (i.e., every $(n-1)$-face is the intersection of exactly two n-faces), then $\mathcal{K}$ is called a **closed** pseudomanifold.

For every n-face F', containing an $(n-2)$-face F, there exists unique edge e, such that $F' = F \cup e$. The **link of an $(n-2)$-face** F is the 1-complex consisting of all edges e as above, where F' run through all n-simplexes, containing F. If the number of such faces F' is at most 5 or if $\mathcal{K}$ is a manifold, then the link consists of a unique cycle. The length of this cycle will be denoted by $l(F)$.

Every compact manifold M can be represented as a closed simplicial complex, if one chooses a triangulation of M.

Definition 1.1: A closed simplicial complex $\mathcal{K}$ is called **of type** L, if for any $(n-2)$-face F of $\mathcal{K}$, one has $l(F) \in L$.

We will be concerned below, especially, with the case $L = \{3, 4\}$.

Some examples:

(i) the boundary of the $(n+1)$-simplex (respectively, the boundary of the $(n+1)$-hyperoctahedron) are examples of simplicial complexes of type $\{3, 4\}$, where, moreover, $L = \{3\}$ (respectively, $L = \{4\}$);

(ii) the only 2-dimensional $\{3, 4\}$-simplicial complexes are: dual Triangular Prism, Tetrahedron and Octahedron.

Take the boundary of the $(n+1)$-hyperoctahedron, which is a simplicial n-complex, and write its set of vertices as $\{1,2,\dots,n+1,1',2',\dots,(n+1)'\}$. This $(n+1)$-hyperoctahedron has the following 2^{n+1} n-faces:

$$\{x_1,\dots,x_{n+1}\} \text{ with } x_i = i \text{ or } i' \text{ for } 1 \le i \le n+1.$$

Definition 1.2: Let $P = (P_1,\dots,P_t)$ be a partition of $V_{n+1} = \{1,2,\dots,n+1\}$. Define the simplicial complex $\mathcal{K}(P)$ as follows:

(i) it has $n+1+t$ vertices $(1,2,\dots,n+1,P_1,\dots,P_t)$,

(ii) every n-face $F_x = \{x_1,\dots x_{n+1}\}$ is mapped onto $F_y = \{y_1,\dots,y_{n+1}\}$, a candidate for an n-face of $\mathcal{K}(P)$, where $y_i = i$ if $x_i = i$, and $y_i = P_j$ if $x_i = i'$, $i \in P_j$.

Below, K_m denotes the complete graph on m vertices, C_m denotes the cycle on m vertices. Denote by $K_m - C_h$ the complement in K_m of the cycle C_h; denote by $K_m - hK_2$ the complete graph on m vertices with h disjoint edges deleted.

Any simplicial or cubical (see Section 3) complex of type $\{3,4\}$ is realizable as a manifold, since the neighborhood of every point is homeomorphic to the sphere.

Given a complex $\mathcal{K}$, its **skeleton** $G(\mathcal{K})$ is the graph with vertices of $\mathcal{K}$ and with two vertices being adjacent if they form an 1-face of $\mathcal{K}$.

Given a graph G, its **path-metric** (denoted by $d_G(i,j)$) between two vertices i, j is the length of a shortest path between them. The graph G is said to be **embeddable up to scale λ into a hypercube** if there exist a mapping ϕ of G into $\{0,1\}^N$ with $\|\phi(i) - \phi(j)\|_{L^1} = \lambda d_G(i,j)$. For the details on such embeddability, see the book [DeLa97].

For example, Proposition 7.4.3 of [DeLa97] gives that $K_{m+1} - K_2$ and $K_{2m} - mK_2$ embed in the $2a_m$-hypercube with a scale $\lambda = a_m$, where $a_m = \binom{m-2}{m/2-1}$ for m even and $a_m = 2\binom{m-2}{(m-3)/2}$ for m odd. Clearly, any subgraph G of $K_{2m} - mK_2$, containing $K_{m+1} - K_2$, also admits the above embedding, since any subgraph of diameter two graph is an **isometric** subgraph. In general, if G is an isometric subgraph of a hypercube, then it is an induced subgraph, but this implication is strict.

A graph is said to be **hypermetric** if its path-metric satisfies the inequality

$$\sum_{1\le i<j\le n} b_i b_j d_G(i,j) \le 0$$

for any vector $b \in \mathbb{Z}^n$ with $\sum_i b_i = 1$. In the special case, when b is a permutation of $(1,1,1,-1,-1,0,\dots,0)$, the above inequality is called **5-gonal**. The validity of hypermetric inequalities is necessary for embeddability but not sufficient: an example of a hypermetric, but not embeddable graph (amongst those, given in Chapter 17 of [DeLa97]) is $K_7 - C_5$.

2. Simplicial complexes of type $\{3,4\}$

In this section, we classify the simplicial complexes of type $\{3,4\}$ in terms of partitions. Let $\mathcal{K}$ be a simplicial complex of type $\{3,4\}$ and let $\Delta = \{1,\dots,n{+}1\}$ be an n-face of this complex. Denote by $F_i = \{1,\dots,i-1,i+1,\dots,n+1\} = V_{n+1} - \{i\}$ an $(n-1)$-face of Δ. F_i is contained in another n-face, which we write as $\Delta_i = \{1,\dots,i-1,i',i+1,\dots,n+1\}$. Denote by $F_{i,j} = V_{n+1} - \{i,j\}$ the $(n-2)$-faces of $\mathcal{K}$. One has $l(F_{i,j}) = 3$ if and only if $i' = j'$. Now, $l(F_{i,j}) = 4$ if and only if $i' \neq j'$ and (i',j') is an edge.

Define a graph on the set V_{n+1} by making i and j adjacent if $l(F_{i,j}) = 3$. By what we already know about i' and j', one obtains that this graph is of the form $K_{P_1} + \cdots + K_{P_t}$ (where K_A denotes the complete graph on the vertex-set A) and so, one gets a **characteristic partition** of V_{n+1}, which we write as $P = \{P_1,\dots,P_t\}$.

THEOREM 2.1: *If $\Delta = \{1,2,\dots,n+1\}$ is a simplex of a simplicial complex $\mathcal{K}$ of type $\{3,4\}$, then $\mathcal{K}$=$\mathcal{K}(P)$ with P being the characteristic partition of Δ.*

Moreover, all simplexes of $\mathcal{K}$ have the same characteristic partition, up to permutations.

Proof: According to the above notation, we define the vertices i' and simplexes Δ_i, such that $\Delta \cap \Delta_i = F_i$. The vertex-set of the complex $\mathcal{K}$ contains vertices $\{1,\dots,n+1,1',\dots,(n+1)'\}$; we will show that it contains no others.

Let us find the values of the link numbers $l(F)$ for an adjacent simplex, say Δ_1 of Δ.

Take an $(n-2)$-face F in $\{1',2,\dots,n+1\}$. If $1' \notin F$, then one has an $(n-2)$-face of Δ and so we already know $l(F)$.

Let us write F as $F'_{i,j} = \{1',2,\dots,n+1\} - \{i,j\}$. The face $F_{i,j}$ is contained in the simplexes Δ, Δ_i and Δ_j. If $l(F_{i,j}) = 3$, then $i' = j'$. If $l(F_{i,j}) = 4$, then $F_{i,j}$ is also contained in $F_{i,j} \cup \{i',j'\}$, which is a simplex of $\mathcal{K}$.

The face $F'_{i,j}$ is contained in the $(n-1)$-faces $\{1',2,\dots,n+1\} - \{i\}$ and $\{1',2,\dots,n+1\} - \{j\}$. According to $l(F_{1,i}) = 3$ or 4, the face $F'_{i,j}$ is contained in either Δ_i (and $i' = 1'$), or in $\{1',2,\dots,i-1,i',i+1,\dots,n+1\}$. The same holds for $F_{1,j}$.

We deal with all cases:

- If $l(F_{i,j}) = 3$, one has $i' = j'$.
  - If $l(F_{1,i}) = 4$ and $l(F_{1,j}) = 4$, then $F'_{i,j}$ is contained in

    $$\{1', 2, \ldots, n+1\}, \quad \{1', 2, \ldots, i-1, i', i+1, \ldots, n+1\} \quad \text{and} \quad \{1', 2, \ldots, j-1, j', j+1, \ldots, n+1\}.$$

    By equality $i' = j'$, one has $l(F'_{i,j}) = 3$.
  - If $l(F_{1,i}) = 3$, then $1' = i'$; so, $1' = j'$ and $l(F_{1,j}) = 3$. The face $F'_{i,j}$ is contained in $\{1', 2, \ldots, n+1\} = F'_{i,j} \cup \{i, j\}$,

    $$\{1', 2, \ldots, i-1, 1, i+1, \ldots, n+1\} = F'_{i,j} \cup \{1, j\}$$

    and $\{1', 2, \ldots, j-1, 1, j+1, \ldots, n+1\} = F'_{i,j} \cup \{i, 1\}$. By equality $i' = j'$, one has $l(F'_{i,j}) = 3$.
- If $l(F_{i,j}) = 4$, one has $i' \neq j'$.
  - If $l(F_{1,i}) = 4$ and $l(F_{1,j}) = 4$, then $F'_{i,j}$ is contained in

    $$\{1', 2, \ldots, n+1\} = F'_{i,j} \cup \{i, j\},$$

    $$\{1', 2, \ldots, i-1, i', i+1, \ldots, n+1\} = F'_{i,j} \cup \{i', j\} \text{ and}$$

    $$\{1', 2, \ldots, j-1, j', j+1, \ldots, n+1\} = F'_{i,j} \cup \{i, j'\}.$$

    Since the length of a link should be 3 or 4 and since we have already 4 vertices, one gets that $F'_{i,j}$ is contained in $F'_{i,j} \cup \{i', j'\}$ and $l(F'_{i,j}) = 4$.
  - If $l(F_{1,i}) = 3$, then $1' = i'$ and so $1' \neq j'$, which implies $l(F_{1,j}) = 4$. The face $F'_{i,j}$ is contained in $\{1', 2, \ldots, n+1\} = F'_{i,j} \cup \{i, j\}$, $\{1', \ldots, i-1, 1, i+1, \ldots, n+1\} = F'_{i,j} \cup \{1, j\}$ and

    $$\{1', 2, \ldots, j-1, j', j+1, \ldots, n+1\} = F'_{i,j} \cup \{i, j'\}.$$

    So, by the same argument, one gets that $F'_{i,j}$ contained in $F'_{i,j} \cup \{1, j'\}$ and $l(F'_{i,j}) = 4$.

One obtains $l(F'_{i,j}) = l(F_{i,j})$. Therefore, Δ and Δ_1 have the same characteristic partition, up to a permutation. Moreover, one sees that the adjacent simplexes to Δ_1 are contained in the vertex-set $\mathcal{V} = \{1, \ldots, n+1, 1', \ldots, (n+1)'\}$. This implies that the vertex-set of $\mathcal{K}$ is exactly $\mathcal{V}$. On the other hand, the characteristic partition of Δ defines uniquely the complex $\mathcal{K}$. Since the complex $\mathcal{K}(P)$ has the same characteristic partition, one obtains the equality $\mathcal{K} = \mathcal{K}(P)$. ∎

Given a complex $\mathcal{K}$, its automorphism group $\mathrm{Aut}(\mathcal{K})$ is defined as the group of permutations of its vertices, preserving the set of faces.

Call a complex **isohedral** if $\mathrm{Aut}(\mathcal{K})$ is transitive on its n-faces.

COROLLARY 2.2: (i) *Given two partitions P and P' of V_{n+1}, one has $\mathcal{K}(P)$ isomorphic to $\mathcal{K}(P')$ if and only if P' is obtained from P by a permutation of V_{n+1}.*

(ii) *Every simplicial complex of type $\{3,4\}$ is isohedral.*

Proof: (i) By Theorem 2.1, all simplexes of a simplicial complex of type $\{3,4\}$ have the same characteristic partition. So, if two complexes of type $\{3,4\}$ are isomorphic, their corresponding partitions are isomorphic too. On the other hand, two isomorphic partitions define the same simplicial complex of type $\{3,4\}$.

(ii) By Theorem 2.1, one can assume that $\mathcal{K}$ is of the form $\mathcal{K}(P)$. Take another n-face $\Delta' = \{v_1,\dots,v_{n+1}\}$ of $\mathcal{K}(P)$; its partition type is the same as of $\{1,\dots,n+1\}$. So, one can construct a mapping φ from $\{1,\dots,n+1\}$ to $\{v_1,\dots,v_{n+1}\}$, preserving partitions and, by extension, being an automorphism of the complex. ∎

One can check that if $\mathcal{K}$ is a n-dimensional simplicial complex of type $\{3,4\}$, such that $l(F)=3$ for each of $(n-2)$-faces F, contained in a fixed $(n-1)$-face, then $\mathcal{K}$ is the boundary of the $(n+1)$-simplex.

THEOREM 2.3: *Every simplicial complex of type $\{3,4\}$ is spherical.*

Proof: Take the partition $\{1\},\dots,\{n+1\}$ of V_{n+1}; the corresponding complex is hyperoctahedron, which is, of course, spherical. Take now a complex $\mathcal{K}(P)$ of type $\{3,4\}$. By merging vertices i' and j', belonging to the same part, we preserve the sphericity. Furthermore, while doing this operation, we do not obtain a pair of different faces having the same set of vertices. So the obtained simplicial complexes are necessarily spherical. ∎

PROPOSITION 2.4: *The skeleton of the simplicial n-complex $\mathcal{K}(P)$ of type $\{3,4\}$ is $K_{n+1+t}-hK_2$, where t is the number of sets of the partition and h is the number of singletons in the partition.*

Proof: In the $(n+1)$-hyperoctahedron, a point i is not adjacent only to the point i'. If i' belongs to a partition of size bigger than 1, then an edge appears; otherwise, there is no such edge. ∎

In particular, K_{n+1} and $K_{2n} - nK_2$ are the skeletons of, respectively, an n-simplex and an n-hyperoctahedron.

THEOREM 2.5: *Every simplicial polytope of the form $\mathcal{K}(P)$ with $P = (P_1, \ldots, P_t)$ is isomorphic to the dual of the product complex $\Delta_1 \times \cdots \times \Delta_t$ with Δ_k being the simplex of dimension $|P_k|$.*

Proof: Let us denote $\mathcal{K} = \Delta_1 \times \cdots \times \Delta_t$; by Corollary 2.2, it suffices to prove that the dual complex $\mathcal{K}^*$ is of type $\{3, 4\}$ and that its characteristic partition is P.

We will argue in dual terms; fix a vertex $1_i \in \Delta_i$ and hence a vertex $v = (1_1, \ldots, 1_t) \in \mathcal{K}$. Any two adjacent vertices in $\mathcal{K}$ differ by exactly one coordinate. Take F a 2-dimensional face of $\mathcal{K}$, which contains v; the vertex v is adjacent to two vertices v_1 and v_2 contained in F. Denote by x_i the coordinates of v_i, which differ from v. If $x_1 = x_2$, then v_1 is adjacent to v_2 and so F has three vertices. If $x_1 \neq x_2$, then F contains the vertex $v' = (1_1, \ldots, (v_1)_{x_1}, \ldots, (v_2)_{x_2}, \ldots, 1_t)$, where $(v_i)_{x_i}$ denotes the x_i-th coordinate of v_i. So F contains the four vertices v, v_2, v', v_1, which form a square. Therefore, $\mathcal{K}^*$ is of type $\{3, 4\}$.

On the other hand, the above computation proves that $\mathcal{K}^*$ is isomorphic to $\mathcal{K}(P)$. ∎

PROPOSITION 2.6: *Every simplicial complex $\mathcal{K}$ of type $\{3, 4\}$ admits a polytopal realization, such that its group of isometries coincides with its group $\mathrm{Aut}(\mathcal{K})$ of combinatorial isometries.*

Proof: By Theorem 2.1, one can assume that $\mathcal{K} = \mathcal{K}(P)$. This result follows immediately from the product decomposition given in Theorem 2.5. ∎

Note that, in general, this group of isometries is a subgroup of $\mathrm{Aut}(\mathcal{K})$. The above proposition is an analog of Mani's theorem ([Ma71]) for 3-connected planar graphs.

PROPOSITION 2.7: *Take a partition $P = (P_1, \ldots, P_t)$ of V_{n+1}; the order of $\mathrm{Aut}(\mathcal{K}(P))$ is*

$$\{\Pi_{k=1}^{t} n_k!\}\Pi_{u=1}^{\infty} m_u!,$$

where $n_k = |P_k| + 1$ and m_u is the number of parts of size u.

Proof: We use again the decomposition given in Theorem 2.5. The symmetry group of the n-simplex has size $(n + 1)!$, which yields the first term of the product. The second term comes from a possible interchange of elements, if sizes of components are equal. ∎

Given a simplicial complex $\mathcal{K}$ of type $\{3,4\}$ and an n-face $\Delta = \{1, \dots, n+1\}$ of $\mathcal{K}$, define g_i to be the "reflection" along the $(n-1)$-face $V_{n+1} - \{i\}$ (i.e., the unique non-trivial automorphism preserving this face). The group, generated by all g_i, is independent on the n-face Δ (because of Proposition 2.8 (iv) below); we will denote it by $\mathrm{Ref}(\mathcal{K})$.

Proposition 2.8: *Given a simplicial complex $\mathcal{K} = \mathcal{K}(P)$ of type $\{3,4\}$, the group $\mathrm{Ref}(\mathcal{K})$ has the following properties:*

(i) $\mathrm{Ref}(\mathcal{K})$ *is a Coxeter group isomorphic to* $\Pi_{k=1}^{t}\,\mathrm{Sym}(n_k)$ *with* $n_k = |P_k|+1$.

(ii) *The generators g_i satisfy the relations*

$$g_i^2 = 1, \quad (g_i g_j)^2 = 1 \quad \text{if } i' \neq j',$$
$$(g_i g_j)^3 = 1 \quad \text{if } i' = j'.$$

But they do not form a simple system *(in terms of [Hu90]) if one has $n_k \geq 4$ for some k.*

(iii) $\mathrm{Ref}(\mathcal{K})$ *is equal to* $\mathrm{Aut}(\mathcal{K})$ *if and only if all parts of the partition have different size (this case includes the simplex and the bipyramid on a simplex).*

(iv) $\mathrm{Ref}(\mathcal{K})$ *is transitive on n-faces.*

(v) *The fundamental domain of* $\mathrm{Ref}(\mathcal{K})$ *is a face if and only if $\mathcal{K}$ is the $(n+1)$-hyperoctahedron (i.e., the action of* $\mathrm{Ref}(\mathcal{K})$ *is regular on the n-faces). If the complex $\mathcal{K}$ is different from the $(n+1)$-hyperoctahedron, then the stabilizer of an n-face in* $\mathrm{Ref}(\mathcal{K})$ *is non-trivial. In general, the fundamental domain of* $\mathrm{Ref}(\mathcal{K})$ *is a simplex with angles π/q for $q = 2$ or 3.*

Clearly, if $\mathcal{K}$ is the boundary of the $(n+1)$-simplex, then $\mathrm{Ref}(\mathcal{K})$ is the irreducible group A_n. For all other simplicial complexes of type $\{3,4\}$, this group is a **reducible** Coxeter group.

The first case, when $\mathrm{Aut}(\mathcal{K})$ is not generated by "reflections" g_i, appears for the complex $\mathcal{K}(\{1,2\},\{3,4\})$. In general, if $\mathrm{Aut}(\mathcal{K})$ of a complex of type $\{3,4\}$ *is* generated by "reflections", then it is a Coxeter group.

Proposition 2.9: *If $\mathcal{K}$ and $\mathcal{K}'$ are two simplicial complexes of type $\{3,4\}$, such that* $\mathrm{Ref}(\mathcal{K})$ *is isomorphic to* $\mathrm{Ref}(\mathcal{K}')$*, then $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic.*

Proof: We express $\mathcal{K}$ (respectively, $\mathcal{K}'$) as $\mathcal{K}(P)$ (respectively, $\mathcal{K}(P')$) and denote by m_u (respectively, by m'_u) the number of parts in P (respectively, P') of size u.

The group $\mathrm{Ref}(\mathcal{K})$ is isomorphic to $(\mathrm{Sym}(2))^{m_1} \times (\mathrm{Sym}(3))^{m_2} \times \cdots$. The group A_k is simple if $k \geq 5$; its multiplicity in the Jordan–Hölder decomposition of $\mathrm{Ref}(\mathcal{K})$ is m_{k-1}. So one has $m_k = m'_k$ if $k \geq 4$. The multiplicities of the cyclic group C_2 (respectively, C_3) in the decomposition of $\mathrm{Ref}(\mathcal{K})$ is $m_1 + m_2 + 3m_3 + \sum_{k\geq 4} m_k$ (respectively, $m_2 + m_3$). One has trivially $n + 1 = \sum_k km_k$. So by solving the linear system, one obtains $m_k = m'_k$ if $k \geq 1$. ∎

A polytope is called **regular-faced** if all its n-faces are regular polytopes.

PROPOSITION 2.10: *Amongst simplicial complexes of type* $\{3, 4\}$*, the only ones admitting regular-faced polytopal realization are two regular ones (the boundary of a simplex and a hyperoctahedron) and the boundary of a bipyramid over a simplex.*

Proof: First, we recall that any simplicial complex of type $\{3, 4\}$ admits a polytopal realization as a convex polytope. All regular-faced polyhedra are known. All 92 3-dimensional ones are classified in [Joh66]. All ones of higher dimension are classified in [BlBl80], [BlBl91] and references 1, 2 therein.

Besides two infinite families (a pyramid over a hyperoctahedron and a bipyramid over a simplex), the list of regular-faced, but not regular, polytopes given in [BlBl91] contains only polytopes in dimension 4. For dimensions 3 and 4, it is easy to check the above proposition. ∎

The following remark, suggested by an anonymous referee, corrects an earlier conjecture of us:

Remark 2.11: In hyperbolic 4-space, there is a regular tessellation by regular 4-simplices, in which the vertex-figures (vertex links) are a regular 600-cell (see [Vin86]). Then it follows from Corollary 4C5 of [MuSe02] that there are infinitely many abstract simplicial 4-complexes of type $\{5\}$.

Hence, there is also an infinite number of 4-complexes of type $\{3, 4, 5\}$. However, if $n = 2$, then, clearly, one obtains only 11 2-complexes on the sphere and 3 2-complexes on the projective plane.

The finiteness of the number of 3-complexes of type $\{3, 4, 5\}$ is expected. However, we have no proof of it and did not do a computer enumeration, since the number of possibilities is too big.

In [Ha00], the n-complexes of type $\{5\}$ are considered; the author found 2 (respectively, 11) complexes of dimension 2 (respectively, 3).

In Table 1, we give details for simplicial complexes of type $\{3, 4\}$ of small dimension. In this Table we mark by $*$ the cases where the group is not Coxeter.

The orbits of vertices are computed with respect to the group $\mathrm{Aut}(\mathcal{K})$.

Two different simplicial complexes of type $\{3,4\}$ with the same skeleton appear, starting from dimension 5: $\mathcal{K}(\{1,2\},\{3,4,5,6\})$ and $\mathcal{K}(\{1,2,3\},\{4,5,6\})$ both have skeleton K_8.

P artition P with $\mathcal{K}=\mathcal{K}(P)$	Skeleton $G(\mathcal{K})$	# n-faces	$\lvert\mathrm{Aut}(\mathcal{K})\rvert$	# orb. vert.	$\lvert\mathrm{Ref}(\mathcal{K})\rvert$
$\{1,2,3\}$	K_4	4	24	1	24
$\{1\},\{2,3\}$	K_5-K_2	6	12	2	12
$\{1\},\{2\},\{3\}$	K_6-3K_2	8	48	1	8
$\{1,2,3,4\}$	K_5	5	120	1	120
$\{1\},\{2,3,4\}$	K_6-K_2	8	48	2	48
$\{1,2\},\{3,4\}$	K_6	9	72^*	1	36
$\{1\},\{2\},\{3,4\}$	K_7-2K_2	12	48	2	24
$\{1\},\{2\},\{3\},\{4\}$	K_8-4K_2	16	384	1	16
$\{1,2,3,4,5\}$	K_6	6	720	1	720
$\{1\},\{2,3,4,5\}$	K_7-K_2	10	240	2	240
$\{1,2\},\{3,4,5\}$	K_7	12	144	2	144
$\{1\},\{2\},\{3,4,5\}$	K_8-2K_2	16	192	2	96
$\{1\},\{2,3\},\{4,5\}$	K_8-K_2	18	144^*	2	72
$\{1\},\{2\},\{3\},\{4,5\}$	K_9-3K_2	24	288	2	48
$\{1\},\{2\},\{3\},\{4\},\{5\}$	$K_{10}-5K_2$	32	3840	1	32

Table 1. All simplicial complexes $\mathcal{K}$ of type $\{3,4\}$ of dimension at most 4

Remark 2.12: The simplicial complex $\mathcal{K}(\{1,2\},\{3,4\})$ has the following properties:

(i) It cannot be realized as a conv ex polytope in $\mathbb{R}^4$, in such a way that each of its 3-faces is *regular* tetrahedron. But this complex admits such embedding in $\mathbb{R}^5$. Moreover, it embeds into a 5-simplex: in fact, into the simplicial complex formed by all 3-dimensional faces of the 5-simplex (apropos, the abov e simplicial complex is not a pseudomanifold).

(ii) It pro vides an example, that the theorem of Alexandrov ([Al50]) does not admits an analog in dimension 3:

An *abstract* n-dimensional Euclidean simplicial complex is formed of simplexes and distances betw een v erticesIf an abstract simplicial complex is realized as the complex formed b y a set of points on the boundary of a polytope (i.e., a *boundary* complex), then it is homeomorphic to an n-sphere and the sum of angles at every vertex is low er thanor equal to the total angle of a sphere of dimension $n-1$ (i.e., it has *non-negative curvature*).

Alexandrov's theorem ([Al50]) asserts that any abstract Euclidean simplicial complex of dimension 2, which is homeomorphic to a 2-sphere and has non-negative curvature, can be realized in $\mathbb{R}^3$ as a boundary complex.

The complex $\mathcal{K}(\{1,2\},\{3,4\})$ has 6 vertices. Let us give equal distances to all edges, i.e., assume that all 3-faces are regular simplexes. It is easy to see that the obtained complex has non-negative curvature. It can be realized in $\mathbb{R}^5$ by the regular 5-simplex. All 4-dimensional polytopes, whose 3-faces are regular 3-simplexes, have been classified in [BlBl80] (see, more generally, Proposition 2.10) and $\mathcal{K}(\{1,2\},\{3,4\})$ is not one of them.

3. Cubical complexes

An n-dimensional **cubical complex** is a lattice, whose n-faces are combinatorial hypercubes. So all its proper faces are combinatorial hypercubes too.

We are interested, especially, in cubical complexes of type $\{3,4\}$.

The hypercubes are only cubical complexes such that any $(n-2)$-face belongs exactly to three n-faces.

The **star** of a vertex in a given complex is the subcomplex formed by all faces which are incident to a given vertex. The star of any vertex of a cubical complex of type $\{3,4\}$ is a **simplicial** complex of type $\{3,4\}$; so the classification of such complexes in Corollary 2.2 characterizes them also, but only locally.

PROPOSITION 3.1: *Let $\mathcal{K}$ be a cubical complex, such that the link of every $(n-2)$-face has size 4; then $\mathcal{K}$ is non-spherical and, moreover:*

(i) *if $\mathcal{K}$ is simply-connected, then it is the cubical lattice Z^n;*

(ii) *otherwise, $\mathcal{K}$ can be obtained as a quotient of Z^n by a torsion-free (i.e., without fixed points) subgroup of the symmetry group of Z^n (i.e., the semidirect product of the Coxeter group B_n and the group of translations).*

Proof: If $\mathcal{K}$ is a cubical complex whose $(n-2)$-faces are contained in exactly four n-faces, then the star of any vertex is $(n-1)$-hyperoctahedron; so one has a unique way to extend it locally to a cubical complex. The simple-connectedness ensures that this construction will not repeat itself. Therefore, one gets the cubical lattice.

If $\mathcal{K}$ is a cubical complex, then its universal cover is the cubical lattice Z^n. So $\mathcal{K}$ is obtained as the quotient of Z^n by a torsion-free subgroup of Aut(Z^n). The sphere is simply-connected; so it cannot be obtained as a proper quotient. ∎

Proposition 3.1(ii) gives, for example, a cubical complex of type $\{4\}$ on the torus and Klein bottle.

Prominent examples (amongst ones given by Proposition 3.1) are *regular toroids* classified in Sections 6D, 6E of [MuSe02].

A 2-dimensional cubical complex is called **quadrillage**. The polytopal quadrillages of type $\{3, 4\}$ are exactly dual **octahedrites**, studied in [DeSt02] and [DDS03], i.e., finite quadrangulations, such that each vertex has valency 3 or 4. So the number of such complexes is not finite already in dimension two. All dual octahedrites, which are isohedral, are: Cube, dual Cuboctahedron and dual Rhombicuboctahedron.

Definition 3.2: Let $\mathcal{K}$ be a cubical complex; define a *zone* as a circuit of $(n-1)$-faces of $\mathcal{K}$, where any two consecutive elements are opposite $(n-1)$-faces of an n-face.

The notion of zone corresponds, in the case of octahedrites, to the notion of **central circuits** (see [DeSt02] and [DDS03]).

4. Embeddability of skeletons of complexes in hypercubes

1267	2346	1246
2367	1456	1357
3467	1237	1234
4567	3457	1345
5167		

The set of 3-faces of the simplicial complex $\mathcal{K}$

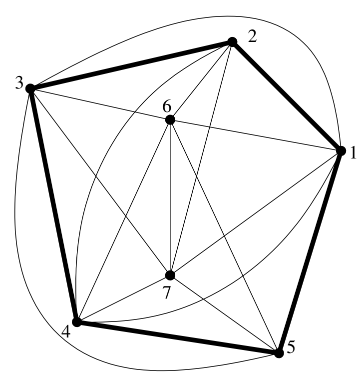

Figure 1. Embeddable triangulation $\mathcal{K}$ with skeleton $K_7 - K_2$ and non-isometric link $C_5 = (1, 2, 3, 4, 5)$ of the edge $(6, 7)$.

THEOREM 4.1: *Let $\mathcal{K}$ be a closed simplicial complex of dimension $n \geq 3$. Then one has:*

(i) *the skeleton of $\mathcal{K}$ is not embeddable, if $\mathcal{K}$ has an $(n-2)$-face belonging to at least five n-simplexes and such that its link is an isometric cycle in the skeleton;*

(ii) *the skeleton of* $\mathcal{K}$ *is embeddable if* $\mathcal{K}$ *is of type* $\{3,4\}$.

Proof: If there exists an $(n-2)$-face, such that its link has size at least six, then the skeleton $\mathcal{K}$ is not 5-gonal, since it contains the isometric subgraph $K_5 - K_3$. If an $(n-1)$-face has a link of size five, then the skeleton of $\mathcal{K}$ contains the isometric subgraph $K_7 - C_5$, which is not embeddable.

All skeletons of simplicial complexes of type $\{3,4\}$ are of the form $K_m - hK_2$ and hence ([DeLa97], Chapter 7.4) embeddable. ∎

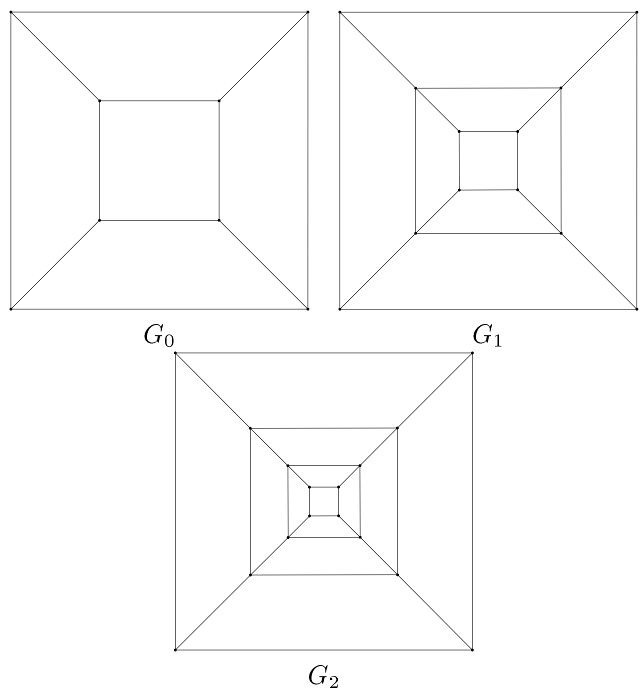

Figure 2. The graphs G_t (embeddable into $(t+3)$-hypercubes) for $t = 0, 1, 2$.

For example, Theorem 4.1(i) implies non-embeddability of the following 3-dimensional simplicial complexes:

(a) a regular 4-polytope 600-cell, since the link of each of its edges is an isometric C_5;

(b) the skeleton of Delaunay partition of the **body-centered cubic lattice** (denoted also A_3^*), since the link of some edges is an isometric C_6 (this skeleton is, moreover, not 5-gonal).

The condition of isometricity of the link in Theorem 4.1 is necessary (it was missed in [DSt98]). For example, there exists an *embeddable* 3-dimensional simplicial complex having an edge which belongs to five tetrahedra; its skeleton is $K_7 - K_2$ (see Figure 1). The same graph $K_7 - K_2$ appears also as the skeleton of a n-dimensional simplicial complex of type $\{3, 4\}$, but only for $n = 4$.

COROLLARY 4.2: *The skeleton of a finite cubical complex is embeddable (moreover, with scale* 1*) if and only if its path-metric satisfies the* 5*-gonal inequalities.*

Proof: Clearly, such skeletons are bipartite graphs, and so the embeddability of the skeleton implies that it is an isometric subgraph of some hypercube or, if infinite, of some cubic lattice Z^m. The result then follows from the characterization of isometric subgraphs of hypercubes, obtained in [Djo73] and reformulated in [Av81]. ∎

On embeddings of **quadrillages** (i.e., cubical complexes of dimension two), we can say more.

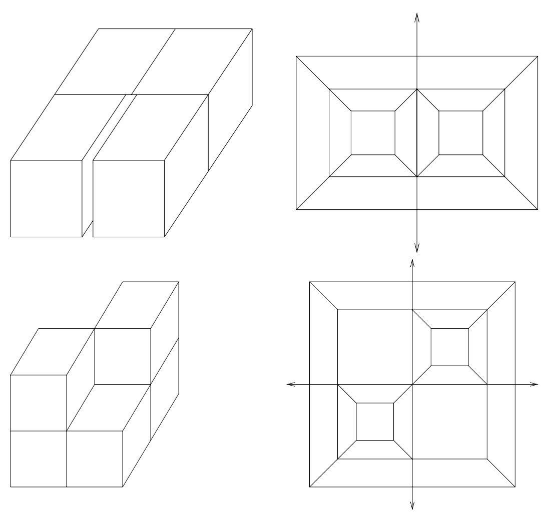

Figure 3. Two non-embeddable quadrillages with simple zones.

One can check that any plane bipartite graph (i.e., all face-sizes are even) is an isometric subgraph of a hypercube if and only if all zones are **simple** (i.e., have no self-intersection) and each of them is convex (i.e., any two of its vertices are connected by a shortest path belonging to the zone). So the skeleton of a quadrillage is embeddable if and only if its zones are convex (and hence simple).

All known embeddable polyhedral quadrillages (i.e., dual octahedrites, which are isometric subgraphs of hypercubes) are zonohedra: dual Cuboctahedron and the family G_t (for any integer $t \geq 0$), illustrated in Figure 2 for the cases $t = 0, 1, 2$. Graphs G_t are embeddable in $(t+3)$-hypercubes.

The simplicity of all zones is a necessary condition for topological embedding of a quadrillage in a cubical lattice Z^m, but it is not sufficient even in the spherical case (see Figure 3). But a quadrillage, such that its skeleton is an *isometric* subgraph of a hypercube, is topologically embeddable in a cubical lattice.

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